International Journal of Heat and Mass Transfer 52 (2009) 1044–1052

Contents lists available at ScienceDirect

International Journal of Heat and Mass Transfer

journal homepage: www.elsevier.com/locate/ijhmt

Field synergy analysis of laminar forced convection between two parallel penetrable walls

Chenhua Gou [a,*], Ruixian Cai [b], Qibin Liu [b]

[a] School of Mechanical and Vehicular Engineering, Beijing Institute of Technology, Beijing 100081, PR China
[b] Institute of Engineering Thermophysics, Chinese Academy of Sciences, P.O. Box 2706, Beijing 100080, PR China

ARTICLE INFO

Article history:
Received 28 November 2006
Received in revised form 17 January 2008
Available online 11 September 2008

Keywords:
Forced convection
Field synergy
Analytical solution

ABSTRACT

In this paper, some exact solutions for 2-D convective heat transfer between two parallel penetrable walls were derived and analyzed based on field synergy theory. They are valuable to further develop the field synergy principle and understand how to improve or to weaken field synergy in practice. In addition, these solutions can be used as benchmarks to verify numerical solutions and to develop numerical schemes, grid generation methods and so forth. All solutions given in this paper can be proven easily by substituting them into the governing equations.

1. Introduction

Analytical analysis, numerical computation and experiment are the three basic ways to research the order of nature. They can work synergically to deepen the understanding of various complex phenomena. The development of field synergy principle for convective heat transfer enhancement is a typical example on hand. In 1998 Guo and his co-workers found out the convection term can be expressed as the dot product of velocity and temperature gradient. And the integral of the dot product over the thermal boundary layer is proportional to the heat transferred by convection for the parabolic fluid flow case [1–3]. Therefore reducing the intersection angle between velocity and temperature gradient can enhance the convective heat transfer effectively. This concept was extended to elliptic flow by both analytical analyses and numerical computations in 2002 [4]. After that, this principle was further confirmed by many numerical and experimental studies [5–14]. All the results led to the establishment of the principle as the unified theory of single phrase convective heat transfer enhancement.

Another example of the "synergy" between different methods is to verify numerical computation methods utilizing exact solutions. It is well known that the exact solutions have their own theoretical meaning. Many exact solutions played key roles in the early development of fluid mechanics and heat conduction [15,16]. Besides their theoretical meaning, exact solutions can also be applied to check the accuracy, convergence and effectiveness of various numerical computation methods and to improve their differencing schemes, grid generation ways and so on. The exact solutions are therefore very useful even for the newly rapidly developing computational fluid dynamics and heat transfer. For instance, several exact solutions which can simulate the 3-D potential flow in turbomachine cascades were obtained by Cai et al. [17], and were successfully used by some investigators in their numerical calculation to check their computational techniques and computer codes [18–21]. In addition, the method of separating variables with addition, which was proposed by the second author, and other special methods were widely applied to different types of equations to derive analytical solutions [22–40].

In this paper, some exact solutions for 2-D convective heat transfer between two parallel penetrable walls were derived and analyzed based on field synergy theory. They are valuable to further develop the field synergy principle and understand how to improve or weaken field synergy in practice.

The derivation of field synergy principle in Ref. [1] was based on the boundary layer energy equation and the integration domain was the thermal boundary layer. The extension of this theory in Ref. [4] was based on the energy equation and the integration domain was the region bounded with solid walls, according to Eq. (6) in [4]. These derivation procedures are independent with any special boundary conditions or special heat transfer mediums. Therefore, we tried to analyze these solutions with the same manner, which makes the results more general. And it's also an advantage of exact solutions over numerical solutions in verifying theories. On the other hand, one of the objects of this paper was to present some physically meaningful benchmark solutions for the computational fluid dynamics and heat transfer. The solutions with better boundary conditions would be more preferable. Therefore, the boundary conditions were also primarily discussed.

* Corresponding author. Tel./fax: +86 10 68911320.
E-mail address: gch6969@yahoo.com.cn (C. Gou).

0017-9310/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijheatmasstransfer.2008.02.052

Nomenclature

a	thermal diffusivity, m^2/s
c_p	specific heat, J/(kg × K)
C_i	arbitrary constant, especially, C_{12} is an arbitrary nonzero integer
$f(y)$	arbitrary function of y
Fc	field synergy number
$g(x)$	arbitrary function of x
h	heat transfer coefficient, $W/(m^2 \times K)$
H	transverse distance between walls, m
Int	The integral defined by Eq. (33)
$k(x, y)$	arbitrary function of x, y
n	arbitrary nonzero integer
Nu	Nusselt number
Nu_0, Nu_1	Nusselt number at $y = 0$ and $y = 1$
p	pressure, Pa
q_0, q_1	heat flux at $y = 0$ and $y = 1$, W/m^2
$r(x)$	arbitrary function of x
T	excess temperature, K
u	velocity component in x direction, m/s
u_m	average velocity in x direction, m/s
U_m	average velocity, m/s
$\overrightarrow{U}$	velocity vector, m/s
$\overline{\overrightarrow{U}}$	dimensionless velocity vector
v	velocity component in y direction, m/s
x, y	x ordinate and y ordinate
$\bar{y}$	dimensionless y ordinate

Greek symbols

η	dynamic viscosity, kg/(m × s)
ρ	density, kg/m^3
$\varphi(y)$	arbitrary function of y
$\psi(x, y)$	arbitrary function of x, y
ΔT	temperature difference between walls, K
ΔNu	total Nusselt number defined by Eq. (32)
$\overrightarrow{\nabla T}$	temperature gradient, K/m
$\overline{\overrightarrow{\nabla T}}$	dimensionless temperature gradient
λ	thermal conductivity, W/(m × K)
θ	field synergy angle between $\overrightarrow{U}$ and $\overrightarrow{\nabla T}$, deg

2. The first solution with uniform *y* direction velocity

The governing equation set of the 2-D steady laminar forced convective heat transfer can be expressed as follows (neglecting gravity and dissipation heat)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \eta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{3}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{4}$$

where T is excess temperature; x and y are geometric coordinates; u and v are corresponding velocity components; p is pressure; ρ is density; a is thermal diffusivity; η is dynamic viscosity. To simplify the analysis, ρ, a, η are assumed to be constant.

Governing equation set (1)–(4) are nonlinear simultaneous partial differential equations. It's not easy to be solved. In order to obtain algebraically explicit analytical solutions for evidently understanding the results and to obtain better benchmark solutions, simple y direction velocity distribution is firstly assumed. The simplest function form of y direction velocity is

$$v = C_1 \tag{5}$$

Substituting Eq. (5) into Eqs. (1) and (3), following results can be deduced

$$\frac{\partial u}{\partial x} = 0, \quad u = f(y) \tag{6}$$

$$\frac{\partial p}{\partial y} = 0, \quad p = g(x) \tag{7}$$

Substituting Eqs. (5)–(7) into Eq. (2), it is derived

$$\rho C_1 f'(y) = -g'(x) + \eta f''(y) \tag{8}$$

The variables can be separated easily

$$\eta f''(y) - \rho C_1 f'(y) = C_2 = g'(x) \tag{9}$$

Then it is deduced

$$f(y) = -C_3 \exp\left(\frac{\rho C_1 y}{\eta}\right) - \frac{C_2 y}{\rho C_1} + C_4 \tag{10}$$

$$g(x) = C_2 x + C_5 \tag{11}$$

Assuming the no slip condition are satisfied at $y = 0$ and $y = 1$

$$u_{y=0} = f(0) = -C_3 + C_4 = 0 \tag{12}$$

$$u_{y=1} = f(1) = -C_3 \exp\left(\frac{\rho C_1}{\eta}\right) - \frac{C_2}{\rho C_1} + C_4 = 0 \tag{13}$$

then following results are derived

$$C_4 = C_3 \tag{14}$$

$$C_2 = -\rho C_1 C_3\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] \tag{15}$$

Therefore, the final solution for the fluid flow Eqs. (1)–(3) is

$$u = -C_3 \exp\left(\frac{\rho C_1 y}{\eta}\right) + C_3\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right]y + C_3 \tag{16}$$

$$v = C_1 \tag{17}$$

$$p = -\rho C_1 C_3\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right]x + C_5 \tag{18}$$

Substituting Eqs. (16) and (17) into the energy equation (4) leads to

$$\left\{-C_3 \exp\left(\frac{\rho C_1 y}{\eta}\right) + C_3\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right]y + C_3\right\}\frac{\partial T}{\partial x} + C_1\frac{\partial T}{\partial y} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{19}$$

Or Eq. (19) can be expressed as

$$\left\{-C_3 \exp\left(\frac{\rho v y}{\eta}\right) - \frac{\partial p}{\partial x}\frac{1}{\rho v}y + C_3\right\}\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{19a}$$

If first assuming

$$T = \varphi(y) \tag{20}$$

Eq. (19) can be simplified as

$$C_1 \frac{d\varphi}{dy} = a \frac{d^2\varphi}{dy^2} \tag{21}$$

Then it is derived

$$T = C_6 + C_7 \exp\left(\frac{C_1 y}{a}\right) \tag{22}$$

Eqs. (16)–(18) and (22) is a solution of the governing equation set (1)–(4) with isothermal boundaries. The wall temperature T_1 (the temperature at $y = 1$) and T_0 (the temperature at $y = 0$) are different. Some constants in this solution are actually physically meaningful. For example, C_1 is the penetration velocity, namely y direction velocity; C_3 reflects the pressure gradient in x direction; and C_7 influences the wall temperature difference proportionally. The boundary conditions could be

$$\begin{array}{ll}
y = 0 & u = 0 \\
 & v = C_1 \\
 & p = -\rho C_1 C_3 \left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] x + C_5 \\
 & T = C_6 + C_7 \\
y = 1 & u = 0 \\
 & v = C_1 \\
 & p = -\rho C_1 C_3 \left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] x + C_5 \\
 & T = C_6 + C_7 \exp\left(\frac{C_1}{a}\right) \\
x = 0 & u = -C_3 \exp\left(\frac{\rho C_1 y}{\eta}\right) + C_3 \left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] y + C_3 \\
 & v = C_1 \\
 & p = C_5 \\
 & T = C_6 + C_7 \exp\left(\frac{C_1 y}{a}\right) \\
x = 1 & u = -C_3 \exp\left(\frac{\rho C_1 y}{\eta}\right) + C_3 \left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] y + C_3 \\
 & v = C_1 \\
 & p = -\rho C_1 C_3 \left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] + C_5 \\
 & T = C_6 + C_7 \exp\left(\frac{C_1 y}{a}\right)
\end{array}$$

The boundary conditions of other solutions given in the following paragraphs can be determined similarly.

The isothermal lines are parallel to x axis. The streamline equation can be derived by $dy/dx = v/u$ and the result is

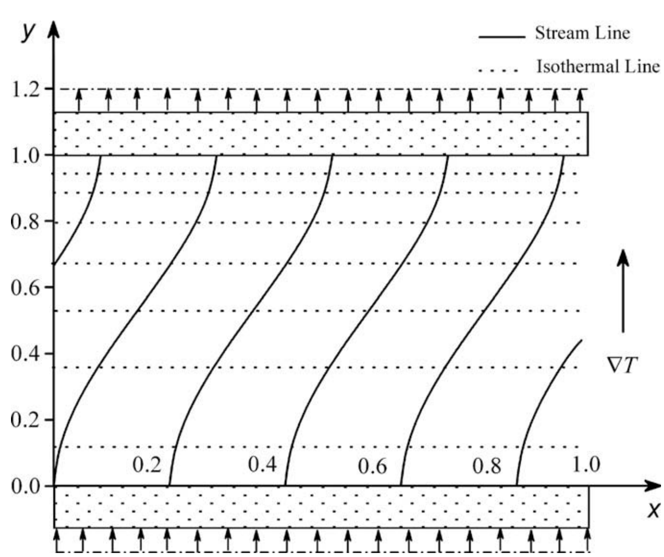

Fig. 1. The stream lines and isothermal lines of the solution Eqs. (16)–(18) and (22) with $C_1 > 0$, $C_3 > 0$ and $C_7 > 0$.

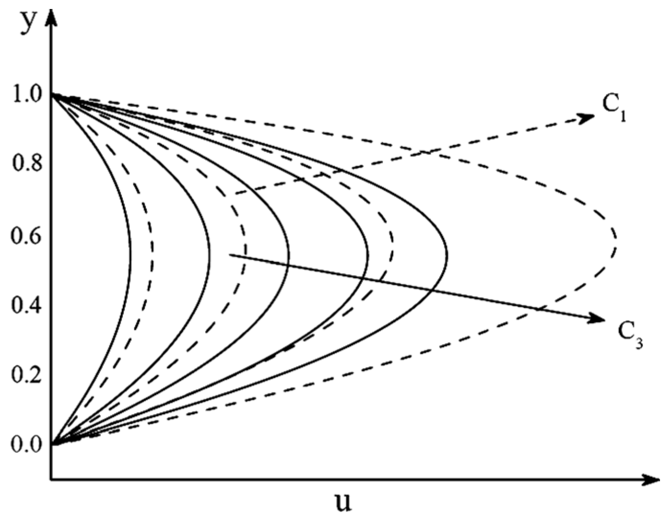

Fig. 2. The variation of u in Eq. (16) with $C_1 > 0$ and $C_3 > 0$.

$$C_1 x = -\frac{\eta C_3}{\rho C_1} \exp\left(\frac{\rho C_1 y}{\eta}\right) + \frac{C_3}{2}\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] y^2 + C_3 y + C_8 \tag{23}$$

When $C_1 > 0$, $C_3 > 0$ and $C_7 > 0$, the stream lines and isothermal lines are illustrated in Fig. 1. The variation of u with C_1 and C_3 is shown in Fig. 2.

The temperature difference between the two walls is

$$\Delta T = T_1 - T_0 = \left[C_6 + C_7 \exp\left(\frac{C_1}{a}\right)\right] - (C_6 + C_7) = C_7\left[\exp\left(\frac{C_1}{a}\right) - 1\right] \tag{24}$$

If assuming

$$C_7 = \left[\exp\left(\frac{C_1}{a}\right) - 1\right]^{-1} \tag{25}$$

it is deduced $\Delta T = 1$, which means the temperature difference between the walls is independent of C_1. If further assuming

$$C_6 = -\left[\exp\left(\frac{C_1}{a}\right) - 1\right]^{-1} \tag{26}$$

Eq. (22) becomes

$$T = -\left[\exp\left(\frac{C_1}{a}\right) - 1\right]^{-1} + \left[\exp\left(\frac{C_1}{a}\right) - 1\right]^{-1} \exp\left(\frac{C_1 y}{a}\right) \tag{27}$$

Then we can get the solution Eqs. (16)–(18) and (27) with $T_1 = 1$ and $T_0 = 0$, which means the wall temperatures are not influenced by the value of C_i. The arbitrary constants C_1 and C_3 can only influence the flow field, but not the thermal boundaries on walls. It is convenient for us to analyze the variation trend of Nu with C_1 and C_3.

The field synergy principle is obviously suitable in this case, since $\frac{\partial T}{\partial x} = 0$, $\int_0^1 \vec{U} \cdot \overrightarrow{\nabla T} dy = a \int_0^1 \frac{\partial^2 T}{\partial y^2} dy$. There is no x direction heat transfer needs to be considered. From Eq. (27) it is deduced

$$h = \frac{\lambda}{(T_1 - T_0)} \frac{\partial T}{\partial y} = \frac{C_1 \lambda}{a \exp\left(\frac{C_1}{a}\right) - a} \exp\left(\frac{C_1 y}{a}\right) \tag{28}$$

The characteristic length for Nusselt numbers is chosen as $2H$, where H is the transverse distance between walls, which equals 1 in this paper. λ is thermal conductivity. Then the expressions of Nusselt numbers are

$$Nu = \frac{2h}{\lambda} = \frac{2C_1}{a\exp\left(\frac{C_1}{a}\right) - a}\exp\left(\frac{C_1 y}{a}\right) \tag{29}$$

$$Nu_0 = \frac{2C_1}{a\exp\left(\frac{C_1}{a}\right) - a} \tag{30}$$

$$Nu_1 = \frac{2C_1}{a\exp\left(\frac{C_1}{a}\right) - a}\exp\left(\frac{C_1}{a}\right) \tag{31}$$

Since the Nusselt numbers on the walls are different, another parameter is defined to facility the analysis

$$\Delta Nu = Nu_1 - Nu_0 = \frac{2}{(T_1 - T_0)}\frac{\partial T}{\partial y}\bigg|_0^1 = \frac{2}{(T_1 - T_0)}\int_0^1 \frac{\partial^2 T}{\partial y^2}\mathrm{d}y = \frac{2C_1}{a} \tag{32}$$

According to the definition of ΔNu, it can be regarded as the total wall Nusselt number. If $q_1 = -\lambda \frac{\partial T}{\partial y}\big|_1$ and $q_0 = -\lambda \frac{\partial T}{\partial y}\big|_0$ have different signs, ΔNu is the dimensionless total wall heat exchange strength. If $q_1 = -\lambda \frac{\partial T}{\partial y}\big|_1$ and $q_0 = -\lambda \frac{\partial T}{\partial y}\big|_0$ are with the same signs, ΔNu is the dimensionless difference value of wall heat exchange strengths. The integral of the dot product of velocity and temperature gradient, which can be regarded as the convective source term, can be expressed as

$$\mathrm{Int} = \int_0^1 \vec{U}\cdot\vec{\nabla}T\mathrm{d}y = \int_0^1 v\frac{\partial T}{\partial y}\mathrm{d}y = C_1(T_1 - T_0) = C_1 \tag{33}$$

It's clear that Int and Nu have nothing to do with C_3 if C_1 is not a function of C_3. The variation of Int, Nu_1, Nu_0 and ΔNu with C_1 is shown in Fig. 3. It is noticeable that the variation of convective source term has quite different influence on the heat transfer condition of different walls. But the variation trends of Int and ΔNu are the same.

The formula of field synergy angle can be deduced as follows

$$\vec{\nabla}T\cdot\vec{U} = \left|\vec{\nabla}T\right|\left|\vec{U}\right|\cos\theta = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = v\frac{\partial T}{\partial y} = \frac{C_1^2 C_7}{a}\exp\left(\frac{C_1 y}{a}\right) \tag{34}$$

$$\left|\vec{\nabla}T\right| = \left|\frac{\partial T}{\partial y}\right| = \frac{C_1 C_7}{a}\exp\left(\frac{C_1 y}{a}\right) \tag{35}$$

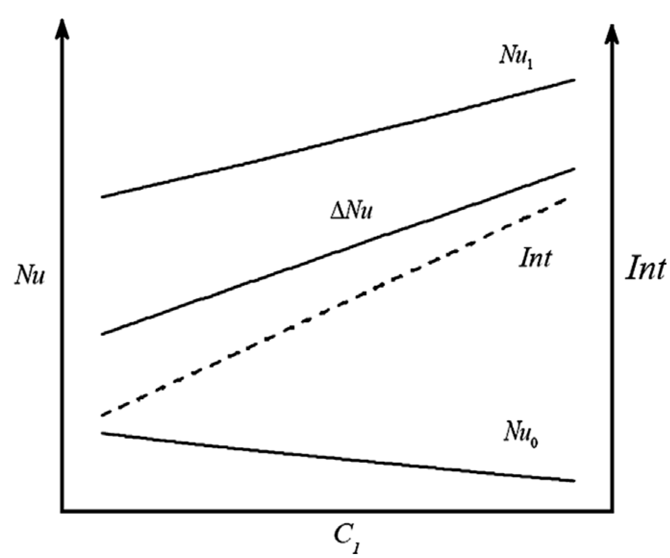

Fig. 3. The variation of Int, Nu_1, Nu_0 and ΔNu with C_1 in the solution Eqs. (16)–(18) and (22).

$$\begin{aligned}\left|\vec{U}\right| &= \sqrt{u^2 + v^2} \\ &= \sqrt{\left\{-C_3\exp\left(\frac{\rho C_1 y}{\eta}\right) + C_3\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right]y + C_3\right\}^2 + C_1^2}\end{aligned} \tag{36}$$

$$\cos\theta = \frac{C_1}{\sqrt{\left\{-C_3\exp\left(\frac{\rho C_1 y}{\eta}\right) + C_3\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right]y + C_3\right\}^2 + C_1^2}} \tag{37}$$

$$\theta = \arccos\frac{C_1}{\sqrt{\left\{-C_3\exp\left(\frac{\rho C_1 y}{\eta}\right) + C_3\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right]y + C_3\right\}^2 + C_1^2}} \tag{38}$$

The variation of θ with C_3 is shown in Fig. 4. It shows θ increases with C_3 which means the degree of field synergy is worse with larger x direction velocity.

The expression of field synergy number can be derived as

$$\mathrm{Fc} = \int_0^1 \overline{\vec{U}}\cdot\overline{\vec{\nabla}T}\mathrm{d}\bar{y} \tag{39}$$

$$\overline{\vec{U}} = \frac{\vec{U}}{U_m}, \quad \overline{\vec{\nabla}T} = \frac{\vec{\nabla}T}{(T_1 - T_0)/1}, \quad \bar{y} = y/1 \tag{40}$$

$$U_m = \sqrt{u_m^2 + v^2} = \sqrt{u_m^2 + C_1^2} \tag{41}$$

$$\begin{aligned}u_m &= \int_0^1\left\{-C_3\exp\left(\frac{\rho C_1 y}{\eta}\right) + C_3\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right]y + C_3\right\}\mathrm{d}y \\ &= \left\{-\frac{C_3\eta}{\rho C_1}\exp\left(\frac{\rho C_1 y}{\eta}\right) + \frac{C_3}{2}\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right]y^2 + C_3 y\right\}\bigg|_0^1 \\ &= -\frac{C_3\eta}{\rho C_1}\exp\left(\frac{\rho C_1}{\eta}\right) + \frac{C_3}{2}\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] + \frac{C_3\eta}{\rho C_1} + C_3\end{aligned} \tag{42}$$

$$\mathrm{Fc} = \int_0^1 \overline{\vec{U}}\cdot\overline{\vec{\nabla}T}\mathrm{d}\bar{y} = \frac{1}{U_m(T_1 - T_0)}\int_0^1 v\frac{\partial T}{\partial y}\mathrm{d}y = \frac{C_1}{U_m} \tag{43}$$

The variation of Fc with C_3 is shown in Fig. 5. Figs. 4 and 5 demonstrate that the field synergy number is a decreasing function, while θ is an increasing function of x direction velocity, when y direction velocity is kept constant. At the same time Int, Nu_1, Nu_0 and ΔNu are

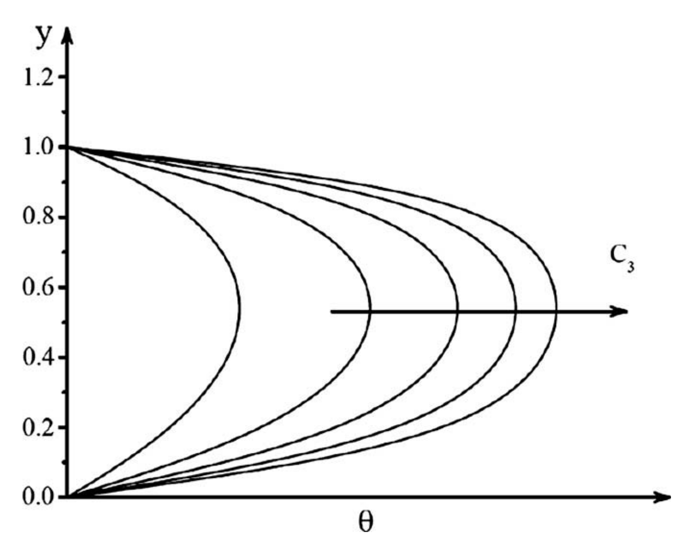

Fig. 4. The variation of θ with C_3 in the solution Eqs. (16)–(18) and (22).

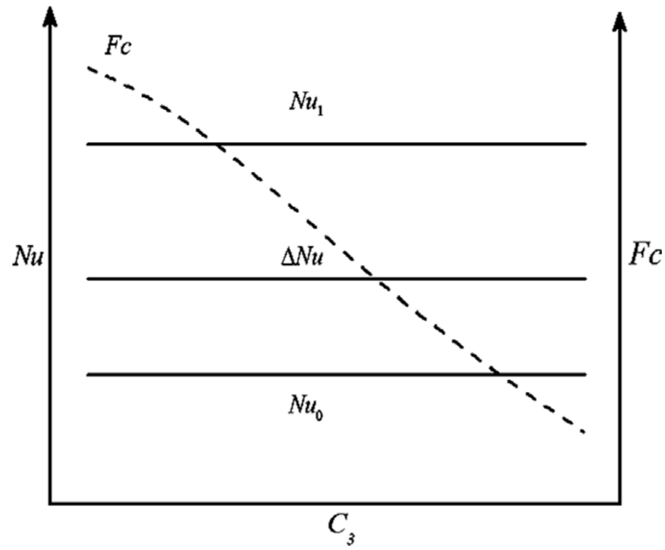

Fig. 5. The variation of Fc with C_3 in the solution Eqs. (16)–(18) and (22).

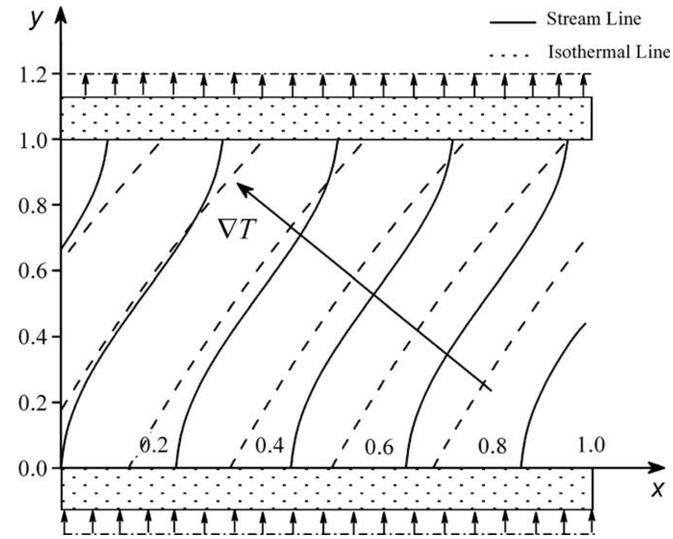

Fig. 6. The stream lines and isothermal lines of the solution Eqs. (16)–(18) and (46) with $C_1 > 0$, $C_3 > 0$, $C_9 > 0$, $a\rho > \eta$ and $C_6 = C_7 = 0$.

not influenced by x direction velocity. Generally, for the convective heat transfer process which satisfies $\frac{\partial T}{\partial x} = 0$ in the whole computation domain, the x direction velocity $u = \psi(x, y)$ might influence Fc and θ but not Nu or Int, which makes the variation trend between them are different.

3. The second solution with uniform y direction velocity

If it is assumed

$$T = \varphi(y) - C_9 x \tag{44}$$

Eq. (19) can be simplified as

$$C_1\frac{d\varphi}{dy} - \left\{-C_3\exp\left(\frac{\rho C_1 y}{\eta}\right) + C_3\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right]y + C_3\right\}C_9 = a\frac{d^2\varphi}{dy^2} \tag{45}$$

C_9 actually could reflect the temperature gradient along the walls. The result is

$$T = C_6 + C_7\exp\left(\frac{C_1 y}{a}\right) + \frac{C_3C_9\eta^2}{a\rho^2C_1^2 - \eta\rho C_1^2}\exp\left(\frac{\rho C_1 y}{\eta}\right) + \frac{C_3C_9}{2C_1}\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right]y^2 + \frac{aC_3C_9\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] + C_1C_3C_9}{C_1^2}y - C_9x \tag{46}$$

Eqs. (16)–(18) and (46) is a solution of the governing equation set (1)–(4). The temperatures on the walls are linear functions of x. The temperature difference between walls is constant.

$$\Delta T = C_7\exp\left(\frac{C_1}{a}\right) + \frac{C_3C_9\eta^2}{a\rho^2C_1^2 - \eta\rho C_1^2}\exp\left(\frac{\rho C_1}{\eta}\right) + \frac{C_3C_9}{2C_1}\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] + \frac{aC_3C_9\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] + C_1C_3C_9}{C_1^2} - \frac{C_3C_9\eta^2}{a\rho^2C_1^2 - \eta\rho C_1^2} - C_7 \tag{47}$$

When $C_1 > 0$, $C_3 > 0$, $C_9 > 0$, $a\rho > \eta$ and $C_6 = C_7 = 0$, the stream lines and isothermal lines are illustrated in Fig. 6.

From Eq. (46) the expression of temperature gradient is

$$\frac{\partial T}{\partial y} = \frac{C_1C_7}{a}\exp\left(\frac{C_1 y}{a}\right) + \frac{C_3C_9\eta}{a\rho C_1 - \eta C_1}\exp\left(\frac{\rho C_1 y}{\eta}\right) + \frac{C_3C_9}{C_1}\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right]y + \frac{aC_3C_9\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] + C_1C_3C_9}{C_1^2} \tag{48}$$

If assuming

$$\left.\frac{\partial T}{\partial y}\right|_{y=y_1} = 0 \tag{49}$$

where y_1 is a constant between 0 and 1, it is deduced

$$\frac{C_1C_7}{a}\exp\left(\frac{C_1 y_1}{a}\right) + \frac{C_3C_9\eta}{a\rho C_1 - \eta C_1}\exp\left(\frac{\rho C_1 y_1}{\eta}\right) + \frac{C_3C_9}{C_1}\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right]y_1 + \frac{aC_3C_9\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] + C_1C_3C_9}{C_1^2} = 0 \tag{50}$$

Further assuming

$$C_7 = \frac{C_{10}C_3C_9a}{C_1^2} \tag{51}$$

Eq. (50) can be simplified as

$$C_{10}\exp\left(\frac{C_1 y_1}{a}\right) + \frac{\eta}{a\rho - \eta}\exp\left(\frac{\rho C_1 y_1}{\eta}\right) + \left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right]y_1 + \frac{a\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] + C_1}{C_1} = 0 \tag{52}$$

The expression of C_{10} is

$$C_{10} = \frac{\frac{\eta}{a\rho - \eta}\exp\left(\frac{\rho C_1 y_1}{\eta}\right) + \left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right]y_1 + \frac{a\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] + C_1}{C_1}}{-\exp\left(\frac{C_1 y_1}{a}\right)} \tag{53}$$

Then Eq. (46) becomes

$$T = C_6 + \frac{C_{10}C_3C_9a}{C_1^2}\exp\left(\frac{C_1 y}{a}\right) + \frac{C_3C_9\eta^2}{a\rho^2C_1^2 - \eta\rho C_1^2}\exp\left(\frac{\rho C_1 y}{\eta}\right) + \frac{C_3C_9}{2C_1}\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right]y^2 + \frac{aC_3C_9\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] + C_1C_3C_9}{C_1^2}y - C_9x \tag{54}$$

In the solution Eqs. (16)–(18), (53) and (54) there is

$$\frac{\partial^2 T}{\partial x^2} = 0 \tag{55}$$

which means the field synergy principle is obviously suitable in this case. No x direction heat transfer needs to be considered. Eq. (4) can be simplified as

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = a\frac{\partial^2 T}{\partial y^2}. \tag{56}$$

If integrating Eq. (56) over $[0, y_1]$ and $[y_1, 1]$, it is derived

$$\int_0^{y_1}\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right)\mathrm{d}y = a\frac{\partial T}{\partial y}\bigg|_0^{y_1} = \frac{1}{\rho c_p}\left(-\lambda\frac{\partial T}{\partial y}_{y=0}\right) \tag{57}$$

$$\int_{y_1}^{1}\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right)\mathrm{d}y = a\frac{\partial T}{\partial y}\bigg|_{y_1}^{1} = -\frac{1}{\rho c_p}\left(-\lambda\frac{\partial T}{\partial y}_{y=1}\right) \tag{58}$$

These equations demonstrate that if there is $1 > y_1 > 0$ at which $\frac{\partial T}{\partial y}\big|_{y=y_1} = 0$, the heat transferred between the fluid flow and the wall $y = 0$ are decided by the convective source term in the domain $[0, y_1]$. And the heat transferred between the fluid flow and the wall $y = 1$ is decided by the convective source term in the domain $[y_1, 1]$. They are not related directly with the field synergy degree in the whole computation domain between the walls, if axial heat transfer can be neglected or there is no axial heat transfer, as in this example. Furthermore, if there are $1 > y_2 > y_1 > 0$ at which $\frac{\partial T}{\partial y}\big|_{y=y_2} = \frac{\partial T}{\partial y}\big|_{y=y_1} = 0$, then the heat transferred at the walls $y = 0$ and $y = 1$ are decided by the field synergy degree in the domain $[0, y_1]$ and $[y_2, 1]$, respectively. The field synergy condition in the domain (y_1, y_2) is not directly related with heat transferred at the walls. This gives the way to find out the heat transfer situation on a specific wall though field analysis. This also points out the relationship between the local field synergy condition and the specific wall heat transfer.

More generally, for a given temperature field $T(x, y)$, if there is a curve $y = y(x)$ on which $\frac{\partial T}{\partial y} = 0$ are always satisfied in the domain $x \in [x_1, x_2]$, as instantiated in Fig. 7, it actually divide the velocity field and the temperature field between $[x_1, x_2]$ into several parts. The energy transferred between fluid flow and the upper wall or the bottom wall can be calculated according to the upper or the bottom convective source terms, respectively. It is interesting to find that in the first example of Ref. [1] (see Fig. 1 and Eq. (2) in [1]), the fields are in fact divided by the imaginary boundary of thermal boundary layer. That is why the integral of the dot product over the thermal boundary layer is proportional to the wall heat flux. And the fluid domain beyond the thermal boundary layer contributes nothing to the heat transfer on the wall.

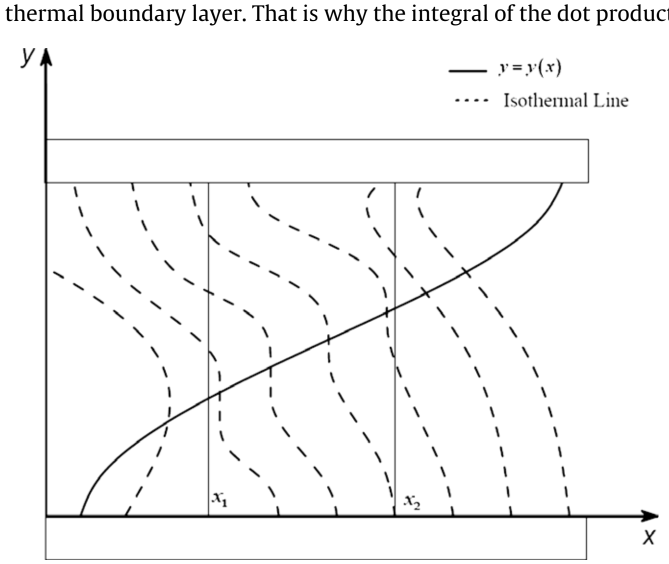

Fig. 7. The example of a curve $y = y(x)$ on which $\frac{\partial T}{\partial y} = 0$ are always satisfied in the domain $x \in [x_1, x_2]$.

The authors notice that in Section 2.2 of Ref. [5], a parallel plate duct with two square insertions was chosen as the computational domain (see Fig. 5 in [5]). The two inserted blocks were assumed to be thermally isolated from the fluid. This example actually can be regarded as a case in which velocity field and temperature field are divided by inserted adiabatic objects. In other words, inserted adiabatic objects can divide fields. Therefore, when considering the duct with inserted objects, some effects need to be taken into account. First, they can introduce interruption within the fluid. Second, they will occupy some space, which decreases the integral domain of convective source term. In addition, if they are adiabatic or can be regarded adiabatic, they can divide the fields.

The variation trend of Int, Nu, θ and Fc with C_i can be analyzed with the same method of the last section.

4. Other possible solutions with uniform *y* direction velocity

Based on the solution Eqs. (16)–(18) and (46), if assuming

$$T_{y=0} = T_{y=1} \tag{59}$$

it is deduced

$$C_7 + \frac{C_3C_9\eta^2}{a\rho^2C_1^2 - \eta\rho C_1^2} = C_7\exp\left(\frac{C_1}{a}\right) + \frac{C_3C_9\eta^2}{a\rho^2C_1^2 - \eta\rho C_1^2}\exp\left(\frac{\rho C_1}{\eta}\right) + \frac{C_3C_9}{2C_1}\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] + \frac{aC_3C_9\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] + C_1C_3C_9}{C_1^2} \tag{60}$$

$$C_7 = \frac{\left\{\frac{C_3C_9\eta^2}{a\rho^2C_1^2 - \eta\rho C_1^2}\exp\left(\frac{\rho C_1}{\eta}\right) + \frac{C_3C_9}{2C_1}\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] + \frac{aC_3C_9\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] + C_1C_3C_9}{C_1^2} - \frac{C_3C_9\eta^2}{a\rho^2C_1^2 - \eta\rho C_1^2}\right\}}{\left[1 - \exp\left(\frac{C_1}{a}\right)\right]} \tag{61}$$

Eqs. (16)–(18), (46) and (61) is a solution of the governing equation set (1)–(4). The temperatures on the walls are linear functions of x. But the wall temperature T_1 and T_0 are equal at the same x.

If assuming

$$\frac{\partial T}{\partial y}_{y=0} = \frac{\partial T}{\partial y}_{y=1} \tag{62}$$

instead Eqs. (60) and (61), we can get

$$\frac{C_7C_1}{a} + \frac{C_3C_9\eta}{a\rho C_1 - \eta C_1} = \frac{C_7C_1}{a}\exp\left(\frac{C_1}{a}\right) + \frac{C_3C_9\eta}{a\rho C_1 - \eta C_1}\exp\left(\frac{\rho C_1}{\eta}\right) + \frac{C_3C_9}{C_1}\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] \tag{63}$$

$$C_7 = \frac{\frac{aC_3C_9\eta}{a\rho C_1^2 - \eta C_1^2}\exp\left(\frac{\rho C_1}{\eta}\right) + \frac{aC_3C_9}{C_1^2}\left[\exp\left(\frac{\rho C_1}{\eta}\right) - 1\right] - \frac{aC_3C_9\eta}{a\rho C_1^2 - \eta C_1^2}}{1 - \exp\left(\frac{C_1}{a}\right)} \tag{64}$$

Eqs. (16)–(18), (46) and (64) is also a solution of the governing equation set (1)–(4). The temperatures on the walls are linear functions of x as well. But the wall heat fluxes are equal at the same x.

5. The solution with variable *y* direction velocity

If the y direction velocity is not uniform in the computation domain, the temperature field and the velocity field would be more complex. It is assumed

$$u = C_{11}\sin^n(C_{12}\pi y)k(x,y) \tag{65}$$

where $k(x,y) \neq \pm\infty$ in the domain $y \in [0,1]$, $x \in [0,1]$; C_{12} and n are arbitrary nonzero integers.

$$\frac{\partial u}{\partial x} = C_{11}\sin^n(C_{12}\pi y)\frac{\partial k(x,y)}{\partial x} \tag{66}$$

According to Eq. (1), it is deduced

$$v = -\int C_{11}\sin^n(C_{12}\pi y)\frac{\partial k(x,y)}{\partial x}\mathrm{d}y \tag{67}$$

For example, assuming $k(x,y) = \exp(C_{13}x)$ and $n = 1$, then we can get

$$u = C_{11}\sin(C_{12}\pi y)\exp(C_{13}x) \tag{68}$$

$$v = \frac{C_{11}C_{13}}{C_{12}\pi}\cos(C_{12}\pi y)\exp(C_{13}x) + r(x) \tag{69}$$

To simplify the derivation, assuming $r(x) = 0$, Eq. (69) becomes

$$v = \frac{C_{11}C_{13}}{C_{12}\pi}\cos(C_{12}\pi y)\exp(C_{13}x) \tag{70}$$

Substituting Eqs. (68) and (70) into Eq. (4) leads to

$$C_{11}\sin(C_{12}\pi y)\exp(C_{13}x)\frac{\partial T}{\partial x} + \frac{C_{11}C_{13}}{C_{12}\pi}\cos(C_{12}\pi y)\exp(C_{13}x)\frac{\partial T}{\partial y} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{71}$$

According to the field synergy principle, reducing the intersection angle between velocity and temperature gradient can enhance the convective heat transfer. On the contrary, increasing the intersection angle between them can weaken the convective heat transfer. Both of these conclusions are useful in practice. Most papers published about field synergy principle are concerning heat transfer enhancement. But here we derive a typical non-synergy explicit solution, which means the intersection angle between velocity and temperature gradient equals $\pi/2$. Hence the heat transfer condition is the worst. It is assumed

$$C_{11}\sin(C_{12}\pi y)\exp(C_{13}x)\frac{\partial T}{\partial x} + \frac{C_{11}C_{13}}{C_{12}\pi}\cos(C_{12}\pi y)\exp(C_{13}x)\frac{\partial T}{\partial y} = 0 = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{72}$$

A special solution of T in the left side of Eq. (72) can be expressed as

$$T = C_{14}\cos(C_{12}\pi y)\exp(C_{13}x) + C_{15} \tag{73}$$

Substituting Eq. (73) into the right side of Eq. (72), it is obtained

$$C_{13} = \pm C_{12}\pi \tag{74}$$

Therefore, Eqs. (68), (70) and (73) can be rewritten as

$$u = C_{11}\sin(C_{12}\pi y)\exp(\pm C_{12}\pi x) \tag{75}$$

$$v = \pm C_{11}\cos(C_{12}\pi y)\exp(\pm C_{12}\pi x) \tag{76}$$

$$T = C_{14}\cos(C_{12}\pi y)\exp(\pm C_{12}\pi x) + C_{15} \tag{77}$$

Substituting Eqs. (75) and (76) into Eqs. (2) and (3), it is deduced

$$p = -\frac{C_{11}^2\rho}{2}\exp(\pm 2C_{12}\pi x) + C_{16} \tag{78}$$

Eqs. (75)–(78) is a non-synergy solution. The y direction velocity boundary conditions are as follows

$$y = 0, \quad u = 0 \tag{79}$$

$$y = 1, \quad u = 0 \tag{80}$$

in addition

$$y = 0, \quad v = \pm C_{11}\exp(\pm C_{12}\pi x) \tag{81}$$

$$y = 1, \text{ when } C_{12} \text{ is an even number}, \quad v = \pm C_{11}\exp(\pm C_{12}\pi x) \tag{82}$$

$$y = 1, \text{ when } C_{12} \text{ is an odd number}, \quad v = \mp C_{11}\exp(\pm C_{12}\pi x) \tag{83}$$

The thermal boundary conditions on the walls are

$$y = 0, \quad q_0 = -\lambda\frac{\partial T}{\partial y} = 0 \tag{84}$$

$$y = 1, \quad q_1 = -\lambda\frac{\partial T}{\partial y} = 0 \tag{85}$$

The x direction boundary conditions are as follows

$$\begin{aligned}
x = 0 \quad & u = C_{11}\sin(C_{12}\pi y) \\
& v = \pm C_{11}\cos(C_{12}\pi y) \\
& p = -\tfrac{C_{11}^2\rho}{2} + C_{16} \\
& T = C_{14}\cos(C_{12}\pi y) + C_{15} \\
x = 1 \quad & u = C_{11}\sin(C_{12}\pi y)\exp(\pm C_{12}\pi) \\
& v = \pm C_{11}\cos(C_{12}\pi y)\exp(\pm C_{12}\pi) \\
& p = -\tfrac{C_{11}^2\rho}{2}\exp(\pm 2C_{12}\pi) + C_{16} \\
& T = C_{14}\cos(C_{12}\pi y)\exp(\pm C_{12}\pi) + C_{15}
\end{aligned}$$

The streamline equation can be derived by $\mathrm{d}y/\mathrm{d}x = v/u$ and the result is

$$\ln|\cos(C_{12}\pi y)| = \mp C_{12}\pi x + C_{17} \tag{86}$$

It is noticeable that in the solution Eqs. (75)–(78)

$$|\vec{U}| = \sqrt{u^2 + v^2} = |C_{11}|\exp(\pm C_{12}\pi x) \tag{87}$$

which has nothing to do with y coordinate. Another feature needs to be pointed out is that the velocity and the temperature distributions is not related with fluid properties. Only the pressure is related with density. Generally, it is easy to understand from Eq. (88) that the temperature distribution has always nothing to do with thermal property a for a non-synergy solution. It is a favorable feature for a benchmark solution.

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = 0 = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), \tag{88}$$

It is also interesting that

$$\frac{\partial T}{\partial x} = \pm C_{12}C_{14}\pi\cos(C_{12}\pi y)\exp(\pm C_{12}\pi x), \tag{89}$$

$$\frac{\partial T}{\partial y} = -C_{12}C_{14}\pi\sin(C_{12}\pi y)\exp(\pm C_{12}\pi x) \tag{90}$$

$$\frac{\partial T}{\partial x}\bigg/\frac{\partial T}{\partial y} = \mp ctg(C_{12}\pi y) \tag{91}$$

$$u/v = \pm tg(C_{12}\pi y) \tag{92}$$

Eqs. (91) and (92) means the internal angles of $\vec{U}$ and $\vec{\nabla}T$ with x axis is independent of x coordinate.

The physical situation when C_{12} is an odd number is different with that when C_{12} is an even number. A typical example when C_{12} is an odd number is as follows, in which $C_{11} = 1$, $C_{12} = 1$, $C_{14} = 1$, $C_{15} = 1$ and $C_{16} = 1$.

$$u = \sin(\pi y)\exp(\pi x) \tag{93}$$

$$v = \cos(\pi y)\exp(\pi x) \tag{94}$$

$$T = \cos(\pi y)\exp(\pi x) + 1 \tag{95}$$

$$p = -\frac{\rho}{2}\exp(2\pi x) + 1 \tag{96}$$

The streamline equations are

$$\ln\cos(\pi y) = -\pi x + C_{17}. \quad y \in [0, 0.5] \tag{97}$$

$$\ln[-\cos(\pi y)] = -\pi x + C_{17}. \quad y \in (0.5, 1] \tag{98}$$

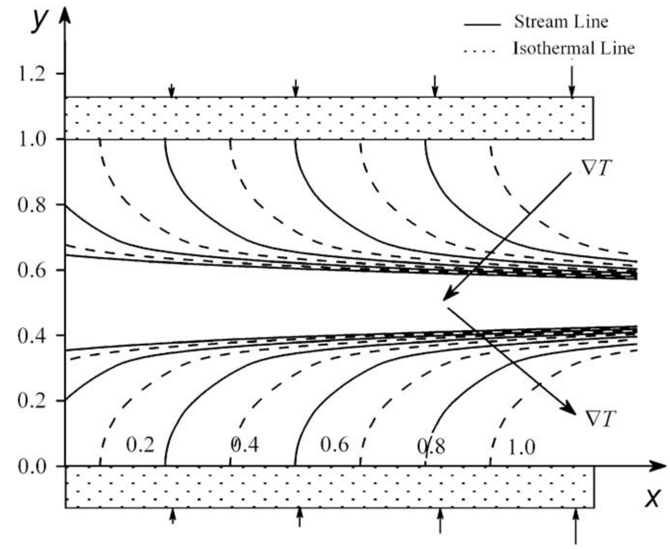

Fig. 8. The stream lines and isothermal lines of the solution Eqs. (93)–(96).

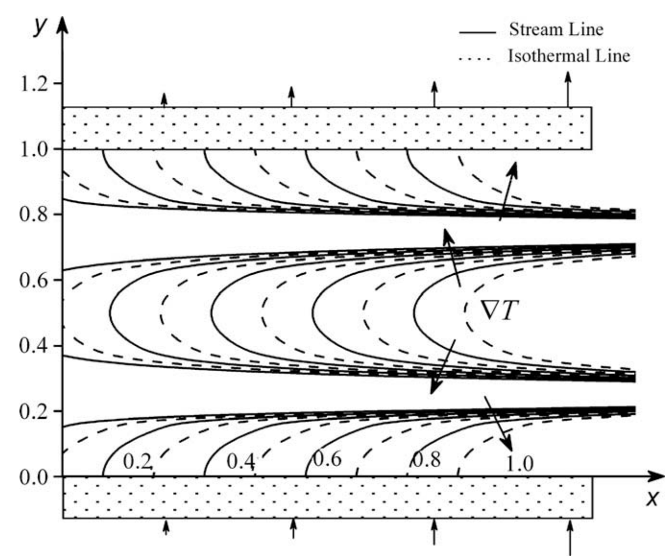

Fig. 9. The stream lines and isothermal lines of the solution Eqs. (99)–(102).

The stream lines and isothermal lines are illustrated in Fig. 8.

A typical example of this solution when C_{12} is an even number is as follows, in which $C_{11}=1$, $C_{12}=2$, $C_{14}=1$, $C_{15}=1$ and $C_{16}=1$.

$$u=\sin(2\pi y)\exp(2\pi x) \tag{99}$$

$$v=\cos(2\pi y)\exp(2\pi x) \tag{100}$$

$$T=\cos(2\pi y)\exp(2\pi x)+1 \tag{101}$$

$$p=-\frac{\rho}{2}\exp(4\pi x)+1 \tag{102}$$

The streamline equations are

$$\ln\cos(2\pi y)=-2\pi x+C_{17}. \quad y\in[0,0.25] \text{ or } y\in[0.75,1] \tag{103}$$

$$\ln[-\cos(2\pi y)]=-2\pi x+C_{17}. \quad y\in(0.25,0.75) \tag{104}$$

The stream lines and isothermal lines are illustrated in Fig. 9. These examples further demonstrate the common belief that the fluid flow enhances the heat transfer is not always true [1]. And one of the methods of enhancing single phrase convective heat transfer, namely increasing the interruption in the fluid are not always effective. They also prove that the fluid motion can be utilized to weaken the heat transfer.

6. Conclusions

Some exact solutions for 2-D convective heat transfer between two parallel penetrable walls were presented in this paper. These results are both theoretically important and valuable to the computational heat transfer as benchmark solutions. And they are valuable to further develop the field synergy principle and understand how to improve or to weaken field synergy in practice. The influence of some factors and the variation of Int, Nu_1, Nu_0, ΔNu, θ and Fc were also discussed. It was demonstrated that the field synergy degree might have different influences on the heat transfer condition of different walls for this kind of flow. It was also pointed out that the local field synergy degree might be more meaningful than the field synergy degree in the whole domain in some cases.

Acknowledgements

Supported by the National Science Foundation of China (Grant No. 50576097) and the National Defense Basic Research Program of China (DEDP 1003).

References

[1] Z.Y. Guo, D.Y. Li, B.X. Wang, A novel concept for convective heat transfer enhancement, Int. J. Heat Mass Transfer 41 (14) (1998) 2221–2225.
[2] S. Wang, Z.X. Li, Z.Y. Guo, Novel concept and device of heat transfer augmentation, in: Proceeding of 11th IHTC, vol. 5, 1998, pp. 405–408.
[3] Z.Y. Guo, S. Wang, Novel concept and approaches of heat transfer enhancement, in: P. Cheng (Ed.), Proceedings of Symposium on Energy and Engineering, vol. 1, Begell House, New York, 2000, pp. 118–126.
[4] W.Q. Tao, Z.Y. Guo, B.X. Wang, Field synergy principle for enhancing convective heat transfer – its extension and numerical verifications, Int. J. Heat Mass Transfer 45 (2002) 3849–3856.
[5] W.Q. Tao, Y.L. He, Q.W. Wang, Z.G. Qu, F.Q. Song, A unified analysis on enhancing single phase convective heat transfer with field synergy principle, Int. J. Heat Mass Transfer 45 (2002) 4871–4879.
[6] J.A. Meng, Enhanced heat transfer technology of longitudinal vortices based on field-coordination principle and its application, Ph.D. Thesis of Tsinghua University, 2003.
[7] S. Shen, W. Liu, W.Q. Tao, Analysis of field synergy on natural convective heat transfer in porous media, Int. Commun. Heat Mass Transfer 30 (2003) 1081–1090.
[8] M. Zeng, W.Q. Tao, Numerical verification of the field synergy principle for turbulent flow, J. Enhanc. Heat Transfer 11 (2004) 451–457.
[9] B. Yu, W.Q. Tao, Pressure drop and heat transfer characteristics of turbulent flow in annular tubes with internal wave-like longitudinal fins, Heat Mass Transfer 40 (2004) 643–651.
[10] Y.P. Cheng, Z.G. Qu, W.Q. Tao, Y.L. He, Numerical design of efficient slotted fin surface based on the filed synergy principle, Numer. Heat Transfer A 45 (2004) 517–538.
[11] Y.L. He, W.Q. Tao*, F.Q. Song, W. Zhang, Three-dimensional numerical study of heat transfer characteristics of plain plate fin-and-tube heat exchangers from view point of field synergy principle, Int. J. Heat Fluid Flow 26 (2005) 459–473.
[12] Z.Y. Guo, W.Q. Tao, R.K. Shah, The field synergy (coordination) principle and its applications in enhancing single phase convective heat transfer, Int. J. Heat Mass transfer 48 (2005) 1797–1807.
[13] Ji-An Meng, Xin-Gang Liang, Zhi-Xin Li, Field synergy optimization and enhanced heat transfer by multi-longitudinal vortexes flow in tube, Int. J. Heat Mass Transfer 48 (2005) 3331–3337.
[14] C.K. Chen, T.S. Yen, Y.T. Yang, Lattice Boltzmann method simulation of a cylinder in the backward-facing step flow with the field synergy principle, Int. J. Therm. Sci. 45 (2006) 982–989.
[15] H. Lamb, Hydrodynamics, sixth ed., Cambridge University Press, London, 1932.
[16] M. Jacob, Heat Transfer, John Wiley & Sons, Inc., New York, 1949.
[17] R. Cai, H. Jiang, C. Sun, Some analytical solutions applicable to verify 3D numerical methods in turbomachines, ImechE Conf. Publ. C80/84 (1984) 255–263.
[18] J. Xu, J. Shi, W. Ni, Three dimensional incompressible flow solution of an axial compressor using pseudostream-functions formulation, ASME Paper 89-GT-319, 1989.

[19] Y. Gong, R. Cai, 3D MSLM – a new engineering approach to the inverse problem of 3D cascade, ASME Paper 89-GT-48, 1989.
[20] M. Shen, Q. Liu, Z. Zhang, Calculation of 3-D transonic flow in turbomachinery with the generalized von Mises coordinates system, Sci. China A 39 (1996) 1084–1095.
[21] R. Cai, Some analytical solutions of unsteady compressible flow, Trans. ASME: J. Fluids Eng. 120 (1998) 760–764.
[22] R. Cai, N. Zhang, Explicit analytical solutions of non-Fourier heat conduction equation for IC chip, Chin. Sci. Bull. 43 (1998) 1080–1084.
[23] R. Cai, N. Zhang, Unsteady 1-D analytical solutions for bio-heat transfer equations, Prog. Nat. Sci. 8 (1998) 733–739.
[24] R. Cai, N. Zhang, Y. He, One-dimensional algebraic explicit solutions of unsteady non-Fourier heat conduction in a sphere, Prog. Nat. Sci. 9 (1999) 34–38.
[25] R. Cai, N. Zhang, Explicit analytical solutions of the coupled differential equations for porous material drying, Prog. Nat. Sci. 10 (2000) 152–157.
[26] R. Cai, N. Zhang, Some algebraically explicit analytical solution of unsteady nonlinear heat conduction, Trans. AMSE; J. Heat Transfer 123 (2001) 1189–1191.
[27] R. Cai, N. Zhang, Explicit analytical solutions of incompressible unsteady 2-D laminar flow with heat transfer, Int. J. Heat Mass Transfer 45 (2002) 2623–2627.
[28] R. Cai, N. Zhang, Explicit analytical solutions of the anisotropic Brinkman model for the natural convection in porous media, Sci. China A 45 (2002) 808–816.
[29] R. Cai, N. Zhang, Explicit analytical solutions of linear and nonlinear interior heat and mass transfer equation sets for drying process, Trans. AMSE: J. Heat Transfer 125 (2003) 175–178.
[30] R. Cai, N. Zhang, Explicit analytical solutions of 2-D laminar natural convection, Int. J. Heat Mass Transfer 46 (2003) 931–934.
[31] R. Cai, N. Zhang, Algebraically explicit analytical solutions of unsteady conduction with variable thermal properties in cylindrical coordinates, Prog. Nat. Sci. 14 (2004) 314–321.
[32] R. Cai, N. Zhang, Algebraically explicit analytical solutions of unsteady conduction with variable thermal properties in spheroidal coordinate, ASME 375 (2004) 137–146.
[33] R. Cai, N. Zhang, C. Gou, Explicit analytical solutions of axisymmetric laminar natural convection along vertical tubes, Prog. Nat. Sci. 15 (2005) 258–261.
[34] C. Gou, R. Cai, N. Zhang, Explicit analytical solutions of transport equations considering non-Fourier and non-Fick effects in porous media, Prog. Nat. Sci. 15 (2005) 545–549.
[35] R. Cai, C. Gou, N. Zhang, Explicit analytical solutions of the anisotropic Brinkman model for the natural convection in porous media (II), Sci. China G 48 (2005) 422–430.
[36] C. Gou, R. Cai, An analytical solution of non-Fourier Chen-Holmes bio-heat transfer equation, Chin. Sci. Bull. 50 (2005) 2791–2792.
[37] C. Gou, R. Cai, One-dimensional unsteady general solution for Wang bioheat transfer equation with heat source proportional to temperature, Prog. Nat. Sci. 15 (2005) 545–549.
[38] R. Cai, C. Gou, H. Li, Algebraically explicit analytical solutions of unsteady 3-D nonlinear non-Fourier (hyperbolic) heat conduction, Int. J. Therm. Sci. 45 (2006) 893–896.
[39] R. Cai, C. Gou, Exact solutions of double diffusive convection in cylindrical coordinates with Le = 1, Int. J. Heat Mass Transfer 49 (2006) 3997–4002.
[40] R. Cai, C. Gou, Discussion on the convective heat transfer and field synergy principle, Int. J. Heat Mass Transfer 50 (2007) 5168–5176.